PMM U.S.S.R.,Vol.46,pp. 356-359 0021-8928/83/3 0356 $7.50/0

UDC 539.3

ON VARIATIONAL INEQUALITY FOR AN OPERATOR OF DYNAMICS OF ELASTIC ROD*

A.M. KHLUDNEV

New formulation of the one-sided problem (variational inequality) for the operator of nonlinear oscillations of a rod with restricted flexure, is given and substantiated. The method can be applied to other nonlinear problems. In the case of a linear operator the method is reduced to the known methods with a given constraint imposed on the velocity of motion.

In recent times a number of results have been obtained for the linear and nonlinear hyperbolic operators, with restriction imposed on the time derivative of the solution. As far as the general hyperbolic operators are concerned, the results obtained apply only to the linear operators /1,2/. In particular /3/ gives the corresponding assertions for the linear theory of elasticity. Concrete nonlinear cases are discussed in /4 – 7/. From the physical point of view however, the most interesting variational inequalities are those in which a constraint is imposed on the function sought /8/ (or, of the parabolic operators are being discussed, on the time integral of the solution, see /9/). Direct application of the penalty operator technique to such problem does not lead to results expected, since the corresponding estimates of the solution are not available.

The operator of nonlinear oscillations of a rod has the form /10/

$$\begin{Vmatrix} v \\ w \end{Vmatrix} \to \begin{Vmatrix} v'' - u_x \\ w'' + w_{xxxx} - (w_x u)_x \end{Vmatrix}, \qquad u = v_x + \frac{1}{2} w_x^2$$

Let $\Omega = (0, l)$, $Q = \Omega \times (0, T)$, $H^m(\Omega) \equiv W_2^m(\Omega)$ be the Sobolev space and $H_0^m(\Omega) \subset H^m(\Omega)$ a subspace obtained by closure of smooth finite functions. The brackets $(\cdot\,,\cdot)$ denote the duality between $H^{-m}(\Omega)$ (conjugated with $H_0^m(\Omega)$) and $H_0^m(\Omega)$, as well as the scalar product in $L^2(\Omega)$. Let K be any closed convex space in $H_0^2(\Omega)$ containing a null element and a set of internal points $\operatorname{int} K$ of which is nonempty.

Theorem. Let $F, G \in L^2(Q)$, $w_0 \in H_0^2(\Omega)$, $v_0 \in H_0^1(\Omega)$, $w_1, v_1 \in L^2(\Omega)$ and $w_0 \in \operatorname{int} K$. Then a pair of functions v, w exists such that

$$v' - \int_0^t u_x\, d\tau = g, \qquad u = v_x + \frac{1}{2} w_x^2 \tag{1}$$

$$\int_0^T \left(w' + \int_0^t \{w_{xxxx} - (w_x u)_x\}\, d\tau - f,\ \varphi - w\right) dt \geqslant 0$$

$$w \in L^\infty(0, T; H_0^2(\Omega)), \qquad v \in L^\infty(0, T; H_0^1(\Omega)),$$

$$w', v' \in L^\infty(0, T; L^2(\Omega))$$

$$g = v_1 + \int_0^t G(\tau)\, d\tau, \qquad f = w_1 + \int_0^t F(\tau)\, d\tau$$

The inequality holds for any $\varphi \in L^2(0, T; H_0^2(\Omega))$, $\varphi(t) \in K$ almost everywhere, and

$$w(0) = w_0,\ v(0) = v_0,\ w(t) \in K \ \text{ almost everywhere} \tag{2}$$

Proof. We introduce the notation

$$w_*(t) = \int_0^t w(\tau)\, d\tau, \qquad v_*(t) = \int_0^t v(\tau)\, d\tau$$

Then (1) and (2) can be rewritten in the form (the asterisk is omitted)

---

*Prikl.Matem.Mekhan.,46,No.3,pp.456-460,1982

$$v'' - \int_0^t U_x \, d\tau = g, \quad U = v_x' + \frac{1}{2} w_x'^2$$

$$\int_0^T \left(w'' + w_{xxxx} - \int_0^t (w_x' U)_x \, d\tau - f, \; \varphi - w' \right) dt \geqslant 0$$

$$w, w' \in L^\infty (0, T; H_0^2 (\Omega)), \quad v, v' \in L^\infty (0, T; H_0^1 (\Omega)),$$
$$w'', v'' \in L^\infty (0, T; L^2 (\Omega))$$

The inequality holds for any $\varphi \in L^2 (0, T; H_0^2 (\Omega))$, $\varphi(t) \in K$ almost everywhere, and $w(0) = 0$, $w'(0) = w_0$, $v(0) = 0$, $v'(0) = v_0$, $w'(t) \in K$ almost everywhere. Let $\beta(\varphi) = J(\varphi - P_k \varphi)$, $J : H_0^2 (\Omega) \to H^{-2} (\Omega)$ be the mapping of the duality and $P_k : H_0^2 (\Omega) \to K$ the projection operator /2/. Consider a problem with a penalty, with fixed $\varepsilon > 0$

$$v_\varepsilon'' - \int_0^t U_{\varepsilon x} \, d\tau = g, \quad U_\varepsilon = v_{\varepsilon x}' + \frac{1}{2} w_{\varepsilon x}'^2 \tag{3}$$

$$w_\varepsilon'' + w_{\varepsilon xxxx} - \int_0^t (w_{\varepsilon x}' U_\varepsilon)_x \, d\tau + \frac{1}{\varepsilon} \beta(w_\varepsilon') = f$$

$$w_\varepsilon(0) = 0, \quad w_\varepsilon'(0) = w_0, \quad v_\varepsilon(0) = 0, \quad v_\varepsilon'(0) = v_0$$

and we show that it has a solution. Let $\{\psi_j\}$ $(j = 1, 2, \ldots)$ be a basis in the space $H_0^2 (\Omega)$. We shall seek the Galerkin approximations to this problem in the form (index ε is omitted)

$$w_n(t) = \sum_{i=1}^n p_{in}(t) \psi_i, \quad v_n(t) = \sum_{i=1}^n q_{in}(t) \psi_i$$

The functions $p_{in}(t)$, $q_{in}(t)$ satisfy the following system of ordinary differential equations:

$$(v_n'', \psi_j) + \left(\int_0^t U_n \, d\tau, \psi_{jx} \right) = (g, \psi_j) \tag{4}$$

$$(w_n'', \psi_j) + (w_{nxx}, \psi_{jxx}) + \left(\int_0^t w_{nx}' U_n \, d\tau, \psi_{jx} \right) + \frac{1}{\varepsilon} (\beta(w_n'), \psi_j) = (f, \psi_j) \tag{5}$$

$$w_n(0) = 0, \; w_n'(0) = w_{n0}, \; v_n(0) = 0, \; v_n'(0) = v_{n0} \tag{6}$$

$w_{n0} \to w_0$ in the norm of $H_0^2 (\Omega)$, $v_{n0} \to v_0$ in the norm of $H_0^1 (\Omega)$. To obtain the a priori estimates for the problem (4)—(6) we first note that if $w_0 \in \operatorname{int} K$, then the approximations w_{n0} will satisfy the same inclusion at sufficiently large n. Thus $\beta(w_{n0}) = 0$ and (4), (5) yield, at $t = 0$ (the constant c_1 is independent of n and $\| \cdot \|$ is the norm in $L^2 (\Omega)$)

$$\| w_n''(0) \| + \| v_n''(0) \| \leqslant c_1$$

Differentiating now (4) and (5) with respect to t, multiplying the equations with index j by $q_{jn}''(t)$, $p_{jn}''(t)$ respectively and summing from 1 to n over j, we obtain

$$\frac{1}{2} \frac{d}{dt} \{ \| w_n''(t) \|^2 + \| w_{nxx}''(t) \|^2 + \| v_n''(t) \|^2 + \| U_n(t) \|^2 \} + \frac{1}{\varepsilon} (\beta(w_n'(t))', w_n''(t)) = (G(t), v_n''(t)) + (F(t), w_n''(t))$$

The expression containing the penalty operator is nonnegative at almost all t, therefore rejecting it and integrating the resulting inequality, we obtain (c_2 is independent of n and the maximum is taken over $t \in [0, T]$)

$$\max \{ \| w_n''(t) \| + \| w_{nxx}'(t) \| + \| v_n''(t) \| + \| U_n(t) \| \} \leqslant c_2 \tag{7}$$

We note that $w_{nx}'^2$ are bounded in $L^\infty (0, T; L^2 (\Omega))$, therefore the boundedness of $\| U_n(t) \|$ implies $\max \| v_{nx}'(t) \| \leqslant c_3$ and hence

$$\max \{ \| w_{nxx}(t) \| + \| v_{nx}(t) \| \} \leqslant c_4 \tag{8}$$

The estimates (7) and (8) guarantee that the system (4), (5) has a solution on $[0, T]$. Taking into account the boundary conditions we conclude, that a subsequence exists, denoted as before by w_n, v_n, with the following properties:

$$\begin{aligned} & w_n \to w, \; w_n' \to w' \; * \text{— weakly in } L^\infty (0, T; H_0^2 (\Omega)) \\ & v_n \to v, \; v_n' \to v' \; * \text{— weakly in } L^\infty (0, T; H_0^1 (\Omega)) \\ & w_n'' \to w'', \; v_n'' \to v'' \; * \text{— weakly in } L^\infty (0, T; L^2 (\Omega)) \end{aligned} \tag{9}$$

Moreover, by virtue of the Lions compactness lemma /2/ we can assume that

$$w_{nx}' \to w_x' \text{ strongly in } L^2 (Q) \tag{10}$$

We shall show that the above convergence is sufficient for the passage to the limit in (4),(5). The properties of the penalty operator imply that

$$\beta(w_n') \to \xi \quad \text{weakly in} \quad L^2(0, T; H^{-2}(\Omega))$$

From (5) we obtain (assuming for the time being that $\varepsilon = 1$)

$$(w_n'', w_n') + (L(w_n, v_n), w_n') + (\beta(w_n'), w_n') = (f, w_n') \tag{11}$$

The symbol $(L(w_n, v_n), w_n')$ denotes the expression obtained from the second and third term of (5), after multiplying by w_n'. Let $\varphi \in L^2(0, T; H_0^2(\Omega))$. We write

$$a_n = \langle(\beta(w_n') - \beta(\varphi), w_n' - \varphi)\rangle \geqslant 0$$

Here and henceforth the angular brackets denote integration with respect to t, from 0 to T. Substituting here $(\beta(w_n'), w_n')$ from (11), we obtain

$$a_n = -\langle(\beta(\varphi), w_n' - \varphi)\rangle - \langle(\beta(w_n'), \varphi)\rangle + \langle(f, w_n')\rangle - \langle(L(w_n, v_n), w_n')\rangle - \langle(w_n'', w_n')\rangle$$

Below we shall show that (see (13))

$$\underline{\lim}\, \langle(L(w_n, v_n), w_n')\rangle \geqslant \langle(L(w, v), w')\rangle$$

therefore

$$0 \leqslant \overline{\lim}\, a_n \leqslant -\langle(\beta(\varphi), w' - \varphi)\rangle - \langle(\xi, \varphi)\rangle + \langle(f, w')\rangle - \langle(L(w, v), w')\rangle - \langle(w'', w')\rangle$$

Taking from the limit equation $w'' + L(w, v) + \xi = f$ the quantity (w'', w') and substituting it into the above expression, we conclude that

$$\langle(\beta(\varphi) - \xi, \varphi - w')\rangle \geqslant 0$$

Let us now take $\varphi = w' + \lambda\psi$, λ — const, $\psi \in L^2(0, T; H_0^2(\Omega))$ arbitrary and utilize the semicontinuity property of the operator β. This yields $\xi = \beta(w')$. Using now (9) and (10), we can pass to the limit in (4) directly, and this proves that the problem (3) has a solution.

The proof shows that the constants c_2 and c_4 in (7),(8) are independent of ε. This means that the functions w_ε and v_ε will satisfy the same inequalities. Therefore a convergence (9), (10) with n replaced by ε exists for the subsequence $w_\varepsilon, v_\varepsilon$. From the second equation of (3) we find that $\beta(w_\varepsilon') \to 0$ in $L^\infty(0, T; H^{-2}(\Omega))$ and also $\langle(\beta(w_\varepsilon'), w_\varepsilon')\rangle \leqslant c_5\varepsilon$, c_5 is independent of ε. The above two relations together with the fact that β is monotonous, yield, in the usual manner, $\beta(w') = 0$, i.e. $w'(t) \in K$ almost everywhere. Let $\varphi \in L^2(0, T; H_0^2(\Omega))$, $\varphi(t) \in K$ almost everywhere, i.e. $\beta(\varphi(t)) = 0$ almost everywhere. The second equation of (3) yields the inequality

$$\left(w_\varepsilon''(t) + w_{\varepsilon xxxx}(t) - \int_0^t (w_{\varepsilon x}' U_\varepsilon)_x\, d\tau - f(t), \varphi(t) - w_\varepsilon'(t)\right) \geqslant 0$$

for nearly all $t \in [0, T]$, therefore

$$\langle(w_\varepsilon'' + w_{\varepsilon xxxx}, \varphi)\rangle + \left\langle\left(\int_0^t w_{\varepsilon x}' U_\varepsilon\, d\tau, \varphi_x\right)\right\rangle - \langle(w_\varepsilon'', w_\varepsilon')\rangle - \langle(f, \varphi - w_\varepsilon')\rangle \geqslant \left\langle\left(w_{\varepsilon xxxx} - \int_0^t (w_{\varepsilon x}' U_\varepsilon)_x\, d\tau, w_\varepsilon'\right)\right\rangle \tag{12}$$

From the weak convergence $w_{\varepsilon x}' v_{\varepsilon x}' \to w_x' v_x'$ in $L^2(Q)$ follows

$$\int_0^t w_{\varepsilon x}' v_{\varepsilon x}'\, d\tau \to \int_0^t w_x' v_x'\, d\tau \quad \text{weakly in } L^2(Q)$$

and, since $w_{\varepsilon x}' \to w_x'$ strongly in $L^2(Q)$, then the passage is possible to the limit in right-hand part of second term (12). The equality

$$\langle(w_{\varepsilon xxxx}, w_\varepsilon')\rangle = \frac{1}{2}\| w_{\varepsilon xx}(T)\|^2$$

holds. From (9) (with index n replaced by ε) we now conclude that $w_\varepsilon(T) \to w(T)$ weakly in $H_0^2(\Omega)$, therefore $\underline{\lim}\, \| w_{\varepsilon xx}(T)\| \geqslant \| w_{xx}(T)\|$. Consequently the lower limit of the right-hand side of (12) is larger than, or equal to

$$\left\langle\left(w_{xxxx} - \int_0^t (w_x' U)_x\, d\tau, w'\right)\right\rangle \tag{13}$$

Passage to the limit in the left-hand side of (12) and in the first equation of (3) as $\varepsilon \to 0$ follows that used in the Galerkin method, and this completes the proof of the theorem.

Introducing minor changes in the proof of the above theorem we can establish a result relevant to the dynamic of an elastoplastic rod. Let the set K correspond to the yield condition

$$K = \{u \in H_0^2(\Omega) \mid |u_{xx}| \leqslant q \quad \text{almost everywhere in} \quad \Omega\},\ q - \text{const} > 0.$$

Then we have the following theorem.

Theorem. Let $F, G \in L^2(\Omega)$, $w_0 \in H^3(\Omega) \cap H_0^2(\Omega)$, $v_0 \in H_0^1(\Omega)$, $w_1, v_1 \in L^2(\Omega)$, with the rod in elastic state at the initial instant, i.e. $|w_{0xx}| < q$. Then the elastoplastic problem (1)—(2) has a solution. The idea of formulating the above variational inequality in the form shown crystallised after the discussions with A.V. Kazhikhov.

REFERENCES

1. BREZIS H.,Problèmes unilatéraux. J. Math. Pures Appl. Vol.51, 1972.
2. LIONS J.L., Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires. Paris, Dunod, 1969.
3. DUVAUT G. and LIONS J.L., Inequalities in Mechanics and Physics. Berlin, Springer-Verlag, 1976. (See also: Les inéquations en mécanique et en physique. Paris, Dunod, 1972).
4. KHLUDNEV A.M., On a onesided problem related to a weakly linear parabolic operator. Sib. matem. zh. Vol.19, No.2, 1978.
5. MAKSUDOV F.G., ALIEV A.B. and TAKHIROV D.M., Onesided problem for a quasilinear, hyperbolic type equation. Dokl. Akad. Nauk SSSR, Vol.258, No.4, 1981.
6. LAR'KIN N.A., On global solutions of nonlinear hyperbolic inequalities. Dokl. Akad. Nauk SSSR, Vol.250, No.3, 1980.
7. KHLUDNEV A.M., On the solvability of the initial boundary value problems for a weakly nonlinear system. Differents. uravneniia, Vol.14, No.11, 1978.
8. KRAVCHUK A.S., On the theory of contact problems taking account of friction on the contact surface. PMM Vol.44, No.1, 1980.
9. KAZHIKHOV A.V., Certain problems of the theory of Navier—Stokes equations for a compressible fluid. In coll.: Dynamics of Continua. Novosibirsk, No.38, 1979.
10. KAUDERER H., Nichtlineare Mechanik. Berlin, Springer, 1958.

Translated by L.K.